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Evaluating Medical College Projects with Hamacher Aggregation Operators under the Interval-valued Complex T-Spherical Fuzzy Environment

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ABSTRACT

The study explores the utilization of Hamacher aggregation operators (HAOs) in inscription selection for medical health projects. We employ interval-valued complex T-spherical fuzzy (IVCTSF) information to address the inherent uncertainties in healthcare data. In this paper, we develop the multiple-attribute decision-making (MADM) problems with IVCTSF set information. A few HAOs built on IVCTSF sets are presented in this work. We practice the Hamacher t-norm (HTNM) and Hamacher t-conorm (HTCNM) to characterize certain operational Hamacher operational rules within the context of the IVCTSF sets. We utilize averaging and geometric operations to develop a family of operators for aggregating IVCTSF information, namely IVCTSF Hamacher weighted averaging (IVCTFHWA), IVCTSF Hamacher order weighted averaging (IVCTSFHOWA), IVCTSF Hamacher hybrid weighted averaging (IVCTSFHHWA), IVCTSF Hamacher weighted geometric (IVCTSFHWG), IVCTSF Hamacher ordered weighted geometric (IVCTSFHOWG), and IVCTSF Hamacher hybrid weighted geometric (IVCTSFHHWG) operators. Several noteworthy properties of the developed operators are examined. Besides, an approach to the MADM algorithm is formulated using the proposed operators and is applied to a detailed case study. The case study measures the effectiveness of the proposed algorithm, analyzes the effect of variable parameters on the decision-making procedure, and ensures the stability of ranking results. A comparative analysis is conducted against existing studies to underscore the significance and advantages. Our findings demonstrate the effectiveness of this approach in improving decision-making for healthcare management in complex scenarios.

1. Introduction

The inherent uncertainty and inaccuracy of data in information analysis have long posed major

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challenges for mathematicians. To mitigate these issues, numerous theoretical frameworks have been proposed, each aiming to enhance the precision and reliability of mathematical modeling and data interpretation. Various theoretical frameworks have been developed to address the frequent errors and inconsistencies encountered in data analysis. These theories are characterized by distinct properties, each offering specific advantages and limitations. Among them, Zadeh's fuzzy set (FS) theory has established itself as a foundational approach. This framework effectively addresses key real-world applications, such as navigation, classification, pattern recognition, and numerous domains in computing and engineering. Zadeh introduced the concept of FS to manage uncertainty by representing membership degrees (MDs) as values within the range $[0, 1]$. This mathematical representation enabled researchers to quantify uncertainty in a structured and formalized manner. However, a fundamental limitation of Zadeh's original formulation was its lack of consideration for a non-membership degree (NMD), which restricted its applicability in certain contexts.

Atanassov [1] enhanced Zadeh's impression of an FS and obtained an intuitionistic fuzzy set (IFS) by presenting MD and NMD to address this need. These ideas demonstrated how ambiguous some limitations can be. This idea set a restriction on the aggregate worth of MD and NMD, preventing it from exceeding 1. Yager [2] enhanced this claim by introducing the idea of the Pythagorean fuzzy set (PyFS), which increases the potential for combining the advantages of MD and NMD. Yager [3] also made another outstanding commitment by inventing the q-rung orthopair fuzzy set (q-ROFS) model. Together, the ideas of uncertainty, PyFS, and q-ROFS tackle practical issues like uncertainty and vulnerability.

The largest flaw, however, was their inability to articulate the specifics of approval and rejection because they had degrees. Because human judgment is not confined to a clear-cut "yes" or "no", it might contain a range of replies. Human evaluations also contain a certain amount of tolerance and resistance. Cuong [4] made an effort to account for this peculiarity. According to him, a significant amount of data is lost when ambiguity is not managed, including its summed-up forms of MD and NMD, promotion, and RD. Cuong proposed the idea of a picture fuzzy set (PFS) as a triad that includes MD, NMD, Promotion, and RD with the constraint that the sum of their values should not exceed 1. To diminish this constraint, Mahmood *et al.* [5] lengthened this notion toward a boundary equal by exemplifying strange spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) sets. The information from the confused plane is not included in the summarized structures outlined above, according to Ramot *et al.* [6]. By using the baffling numbers rather than the real numbers, Ramot *et al.* [6] considered including the perplexing plane FS and offered the prospect for complex FS (CFS).

1.1. Literature Review

The CFS's notion had pushed the FS to its breaking point, but the CFS could not possibly be concerned about the numerous perplexing numbers in the unit circle. To provide leaders with a massive platform to get rid of the greater data when compared to the CFS, Alkouri *et al.* [7] built the possibility of bewildering uncertainties (CIFS). To promote CIFSs, Alkouri and Salleh [7] employed the MD and NMD as a baffling sum after the unit circle on a confusing plane. The MD and NMD of the numbers in the CIFS, however, were represented as a sum of the genuine and fake parts in a unit circle. The issue developed, nevertheless, when leaders selected levels of fantastical and real elements whose aggregate was greater than a unit circle. Ullah *et al.* [8] enclosed more data than the CIFSs in their presentation of the complicated PyFSs (CPyFSs) by increasing the number of degrees to the number of their squares.

The most advanced strategy involves choosing the best option from a minor quantity of explicit options created on a sum of frequently at odds with each other criteria. When the aforementioned systems were enhanced, the MADM technique became very popular since the outcomes it generates depend on the most powerful aggregation operators (AOs). Khan *et al.* [9] highlighted the AOs on uncertainty and used them in MADM. Using the PyFSs, Liu and Wang [10] created AOs that were later incorporated into MADM. As part of their contribution to the MADM, Wang *et al.* [11] created AOs for the q-ROFS premise. Garg [12] implemented many AOs in MADM in light of PFSs. Zhou *et al.* [13] provided AOs for MADM using the TSFS data. There is some outstanding effort on the AOs in [14-16].

A few important functional regulations are required for the creation of these AOs. These regulations rely on a few three-sided criteria to achieve adaptation [17]. Wu *et al.* [18] created Dombi AOs by applying the Dombi t-norm (TNM) and t-conorm (TCNM) in the MADM while taking uncertainties into account. Akram *et al.* [19] framed the Dombi AOs and applied them to PyFS to overcome the MADM issue. Wang and Liu [20] used the Einstein TNM and TCNM to create the AOs for the climate of uncertainty, which were subsequently implemented in MADM. Riaz *et al.* [21] established AOs and provided a beneficial application in the production network of the board by incorporating Einstein TNM and TCNM for the atmosphere of q-ROFSs. Fahmi *et al.* [22] utilized Einstein TNM and TCNM to enhance the AOs for the submission of MADM. Senapati *et al.* [23] suggested the Aczel-Alsina AOs in the context of IFSs. Yang *et al.* [24] a few long-standing PyFS AOs were promoted in preparation for TCNM and the impending TNM. Several AOs that rely on different additional TNMs and TCNMs are mentioned in [25-26].

The AOs that have a significant influence on the request in MADM are developed using the TNMs and TCNMs that were previously stated. Between these TNMs and TCNMs, the HTNM and HTCNM stand out and have been extensively used by professionals in nearly all of the FS hypothesis models that have been created [27]. Garg [28] formalized AOs and applied HTNM and HTCNM to uncertainty. Wu and Wei [29] also used the HTNM and HTCNM in the formalization of the AOs for the PyFS. For the q-ROFS, Darko and Liang [30] provided a few AOs that used HTNM and HTCNM. To evaluate a project, Ullah *et al.* [31] presented AOs for the TSFS based on HTNM and HTCNM. For IFSs, the condensed AOs were provided in [32]. PyFS AOs were developed and used by Wu and Wei [33] with the direction of HTNM and HTCNM. Ali *et al.* [34] fostered some AOs under the environment of complex interval-valued PyFS.

It has long been acknowledged that the CTSFS will cover the significant loss of data when we remove data from any real distinctiveness to perform navigation. In particular, it is incredibly possible to extract the most likely facts whenever the human perspective is taken into account. So, employing CTSFS in MADM provides a wonderful opportunity to engage with the outcomes in the MADM. We further noted the importance of HTNM and HTCNM in the work by Klement and Navara [35] who analyzed various TN and TCN kinds, attained a range of ranks, and found the critical ramifications for HTNM and HTCNM.

1.2. Contribution

The key contributions of this study are described as follows:

i. This paper introduces novel Hamacher operational laws based on IVCTSFs and applies these operations to develop the IVCTSFHWA, IVCTSFHOWA, IVCTSFHHWA, IVCTSFHWG, IVCTSFHOWG, and IVCTSFHHWG operators.

ii. This research examines several key mathematical properties of the proposed operators to validate their effectiveness and theoretical robustness.
iii. The study demonstrates the application of the proposed methods in MAGDM problems using the proposed operators.
iv. A case study is presented to illustrate the practical benefits of the proposed approach, focusing on the evaluation and government development of a medical college.
v. To authenticate the efficiency and robustness of the proposed MADM scheme, extensive sensitivity analysis and a comparative evaluation against existing approaches are conducted, highlighting its superiority and resilience.

This article is planned as follows: Section 2 offers several vital notions related to IVCTSFS, score function, and Hamacher operations. In Section 3, we created the Hamacher operational laws for IVCTSFSs. In Section 4, we studied a series of averaging operators for IVCTSFSs, including the IVCTSFHWA, IVCTSFHOWA, and IVCTSFHHWA operators along with their fundamental properties. In Section 5, we constructed the IVCTSFHWG, IVCTSFOWG, and IVCTSFHHWG operators as well as explored their features. Section 6 presents the development of a MADM methodology utilizing the proposed operators. To demonstrate the practical applicability of the developed framework, an illustrative case study is conducted. In Section 7, a comparative and sensitivity analysis is performed to validate the effectiveness and robustness of the proposed approach. Finally, Section 8 outlines the key conclusions and suggests directions for future research.

2. Preliminaries

In this segment, we characterized fundamental ideas connected to IVCTSFS, HTNM, and HTCNM.

Definition 1: [36] The IVCTSFS over a fixed set X can be defined as:

$$\boldsymbol{I} = \begin{pmatrix} \left[\mathbb{m}_a(\mathfrak{X}).\, e^{2\pi i\vartheta_a(\mathfrak{X})}, \mathbb{u}_a(\mathfrak{X}).\, e^{2\pi i\beta_a(\mathfrak{X})}\right], \\ \left[\mathbb{m}_i(\mathfrak{X}).\, e^{2\pi i\vartheta_i(\mathfrak{X})}, \mathbb{u}_i(\mathfrak{X}).\, e^{2\pi i\beta_i(\mathfrak{X})}\right], \\ \left[\mathbb{m}_n(\mathfrak{X}).\, e^{2\pi i\vartheta_n(\mathfrak{X})}, \mathbb{u}_n(\mathfrak{X}).\, e^{2\pi i\beta_n(\mathfrak{X})}\right] \end{pmatrix}; \mathfrak{X} \in \boldsymbol{X}, \tag{1}$$

with $0 \le \mathbb{m}_a^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_i^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_n^{\mathbb{Q}}(\mathfrak{X}) \le 1$ and $0 \le \vartheta_a^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_i^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_n^{\mathbb{Q}}(\mathfrak{X}) \le 1$ for $\mathbb{Q} \in \mathbb{Z}^+$. The RD can be defined as:

$$\boldsymbol{\pi}(\mathfrak{X}) = \mathbb{m}_h(\mathfrak{X}).\, e^{2\pi i\vartheta_h(\mathfrak{X})}, \tag{2}$$

where $\mathbb{m}_h(\mathfrak{X}) = \sqrt[\mathbb{Q}]{1 - \left(\mathbb{m}_a^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_i^{\mathbb{Q}}(\mathfrak{X}) + \mathbb{m}_n^{\mathbb{Q}}(\mathfrak{X})\right)}$ and $\vartheta_h(\mathfrak{X}) = \sqrt[\mathbb{Q}]{1 - \left(\vartheta_a^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_i^{\mathbb{Q}}(\mathfrak{X}) + \vartheta_n^{\mathbb{Q}}(\mathfrak{X})\right)}$. For simplicity, the triplet

$$\left(\left[\mathbb{m}_a(\mathfrak{X}).\, e^{2\pi i\vartheta_a(\mathfrak{X})}, \mathbb{u}_a(\mathfrak{X}).\, e^{2\pi i\beta_a(\mathfrak{X})}\right], \left[\mathbb{m}_i(\mathfrak{X}).\, e^{2\pi i\vartheta_i(\mathfrak{X})}, \mathbb{u}_i(\mathfrak{X}).\, e^{2\pi i\beta_i(\mathfrak{X})}\right], \left[\mathbb{m}_n(\mathfrak{X}).\, e^{2\pi i\vartheta_n(\mathfrak{X})}, \mathbb{u}_n(\mathfrak{X}).\, e^{2\pi i\beta_n(\mathfrak{X})}\right]\right), \tag{3}$$

is labeled as an IVCTSFN.

Definition 2: [36] For an IVCTSF I, the score value $\mathbb{S}(I) \in [-1, 1]$ is articulated as:

$$\mathbb{S}(\boldsymbol{I}) = \frac{\left((\mathbb{m}_a + \boldsymbol{\vartheta}_a + \mathbb{u}_a + \boldsymbol{\beta}_a) - (\mathbb{m}_i + \boldsymbol{\vartheta}_i + \mathbb{u}_i + \boldsymbol{\beta}_i) + (\mathbb{m}_n + \boldsymbol{\vartheta}_n + \mathbb{u}_n + \boldsymbol{\beta}_n)\right)}{\boldsymbol{6}}. \tag{4}$$

For two IVCTSFNs I_1 and I_2, we can describe the following ordered relations as follows:

i. When $\mathbb{S}(I_1) < \mathbb{S}(I_2)$, then $I_1 < I_2$.
ii. When $\mathbb{S}(I_1) > \mathbb{S}(I_2)$, then $I_1 > I_2$.
iii. When $\mathbb{S}(I_1) = \mathbb{S}(I_2)$, then $I_1 = I_2$.

Definition 3: [27] The HTCNM and HTNM functions can be defined as:

$$T_{hn}(\mathbb{m}, a) = \frac{\mathbb{m}.a}{\mathbb{N} + (1-\mathbb{N})(\mathbb{m} + a - \mathbb{m}a)}, \ \mathbb{N} > 0, (\mathbb{m}, a) \in [0,1]^2, \tag{5}$$

$$T_{hcn}(\mathbb{m}, a) = \frac{\mathbb{m} + a - \mathbb{m}a - (1-\mathbb{N})\mathbb{m}a}{1-(1-\mathbb{N})\mathbb{m}a}, \ \mathbb{N} > 0, (\mathbb{m}, a) \in [0,1]^2. \tag{6}$$

3. Hamacher Operational Rules for IVCTSFNs

In this section, we shall provide the concepts of Hamacher operational laws, including sum, product, scalar multiplication, and scaler power which can be applied to aggregate IVCTSFNs. We have also looked at and examined several fundamental properties.

Definition 4: Let A and B be two IVCTSFNs and $B, \mathbb{N} > 0$ be any real numbers. Then, the Hamacher operations for IVCTSFNs are described as:

$$A \oplus B = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[\mathbb{Q}]{\frac{\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x}) + \mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - \mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - (1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x}) + \vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - \vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - (1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}}, \\ \sqrt[\mathbb{Q}]{\frac{\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x}) + \mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - \mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - (1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x}) + \beta^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - \beta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{a_B}(\mathfrak{x}) - (1-\mathbb{N}).\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{a_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}} \end{array} \right], \\ \left[\begin{array}{c} \frac{\mathbb{m}_{i_A}(\mathfrak{x}).\mathbb{m}_{i_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x}) + \mathbb{m}^{\mathbb{Q}}_{i_B}(\mathfrak{x}) - \mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{i_B}(\mathfrak{x})\right)}} . e^{2\pi i \frac{\vartheta_{i_A}(\mathfrak{x}).\vartheta_{i_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x}) + \vartheta^{\mathbb{Q}}_{i_B}(\mathfrak{x}) - \vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{i_B}(\mathfrak{x})\right)}}}, \\ \frac{\mathbb{u}_{i_A}(\mathfrak{x}).\mathbb{u}_{i_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x}) + \mathbb{u}^{\mathbb{Q}}_{i_B}(\mathfrak{x}) - \mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{i_B}(\mathfrak{x})\right)}} . e^{2\pi i \frac{\beta_{i_A}(\mathfrak{x}).\beta_{i_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\beta^{\mathbb{Q}}_{i_A}(\mathfrak{x}) + \beta^{\mathbb{Q}}_{i_B}(\mathfrak{x}) - \beta^{\mathbb{Q}}_{i_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{i_B}(\mathfrak{x})\right)}}} \end{array} \right], \\ \left[\begin{array}{c} \frac{\mathbb{m}_{n_A}(\mathfrak{x}).\mathbb{m}_{n_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x}) + \mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x}) - \mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}} . e^{2\pi i \frac{\vartheta_{n_A}(\mathfrak{x}).\vartheta_{n_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x}) + \vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x}) - \vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}}, \\ \frac{\mathbb{u}_{n_A}(\mathfrak{x}).\mathbb{u}_{n_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x}) + \mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x}) - \mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}} . e^{2\pi i \frac{\beta_{n_A}(\mathfrak{x}).\beta_{n_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N} + (1-\mathbb{N})\left(\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x}) + \beta^{\mathbb{Q}}_{n_B}(\mathfrak{x}) - \beta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}} \end{array} \right] \end{array} \right), \tag{7}$$

$$A \otimes B = \begin{pmatrix} \begin{bmatrix} \dfrac{\mathbb{m}_{a_A}(\mathfrak{x}).\mathbb{m}_{a_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N}+(1-\mathbb{N})\left(\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})+\mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x})-\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}.e^{2\pi i\frac{\vartheta_{a_A}(\mathfrak{x}).\vartheta_{a_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N}+(1-\mathbb{N})\left(\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})+\vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x})-\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}}, \\ \dfrac{\mathbb{u}_{a_A}(\mathfrak{x}).\mathbb{u}_{a_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N}+(1-\mathbb{N})\left(\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})+\mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x})-\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}.e^{2\pi i\frac{\beta_{a_A}(\mathfrak{x}).\beta_{a_B}(\mathfrak{x})}{\sqrt[\mathbb{Q}]{\mathbb{N}+(1-\mathbb{N})\left(\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})+\beta^{\mathbb{Q}}_{a_B}(\mathfrak{x})-\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{a_B}(\mathfrak{x})\right)}}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[\mathbb{Q}]{\dfrac{\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})+\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-(1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})\right)}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})+\vartheta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-(1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x}2)}{1-\left((1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})\right)}}}, \\ \sqrt[\mathbb{Q}]{\dfrac{\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})+\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-(1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})\right)}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})+\beta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})-(1-\mathbb{N}).\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x}2)}{1-\left((1-\mathbb{N}).\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{\mathrm{i}_B}(\mathfrak{x})\right)}}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[\mathbb{Q}]{\dfrac{\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})+\mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x})-\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x})-(1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{m}^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})+\vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x})-\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x})-(1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\vartheta^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}}, \\ \sqrt[\mathbb{Q}]{\dfrac{\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})+\mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x})-\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x})-(1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\mathbb{u}^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})+\beta^{\mathbb{Q}}_{n_B}(\mathfrak{x})-\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{n_B}(\mathfrak{x})-(1-\mathbb{N}).\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{n_B}(\mathfrak{x})}{1-\left((1-\mathbb{N}).\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x}).\beta^{\mathbb{Q}}_{n_B}(\mathfrak{x})\right)}}} \end{bmatrix} \end{pmatrix}, \quad (8)$$

$$BA = \begin{pmatrix} \begin{bmatrix} \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B-\left(1-\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}}, \\ \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B-\left(1-\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}} \end{bmatrix}, \\ \begin{bmatrix} \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{m}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{\mathrm{i}_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\vartheta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}}}, \\ \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{u}^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{\mathrm{i}_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\beta^{\mathbb{Q}}_{\mathrm{i}_A}(\mathfrak{x})\right)^B}}} \end{bmatrix}, \\ \begin{bmatrix} \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{n_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{n_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}}, \\ \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{n_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{n_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}} \end{bmatrix} \end{pmatrix}, \quad (9)$$

$$A^B = \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{a_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{m}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{a_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\vartheta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}}, \\ \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{a_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\mathbb{u}^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{a_A}(\mathfrak{x})\right)^B}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)\right)^B+(\mathbb{N}-1)\left(\beta^{\mathbb{Q}}_{a_A}(\mathfrak{x})\right)^B}}} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B-\left(1-\vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}}}, \\ \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B-\left(1-\beta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{i_A}(\mathfrak{x})\right)^B}}} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{m}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B-\left(1-\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\vartheta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}}, \\ \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B-\left(1-\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\mathbb{u}^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}} . e^{2\pi i \sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B-\left(1-\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}{\left(1+(\mathbb{N}-1)\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B+(\mathbb{N}-1)\left(1-\beta^{\mathbb{Q}}_{n_A}(\mathfrak{x})\right)^B}}} \end{array} \right] \end{array} \right). \tag{10}$$

4. Averaging Aggregation Operators of IVCTSFNs

In this part, using Hamacher operational laws, we construct a series of averaging AOs for IVCTSFNs, namely IVCTSFHWA, IVCTSFHOWA, and IVCTSFHHWA operators. We also highlight the desirable characteristics and of these AOs with concrete illustrations. Throughout, this script $\Gamma_{\Xi} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_{\mathbb{m}})^T$ will denote the weight vector, such that $\Gamma_{\Xi} > 0$ and $\sum_1^{\mathbb{m}} \Gamma_{\Xi} = 1$ for $\Xi = \{1, 2, 3 \ldots \mathbb{m}\}$.

Definition 5: Let $\mathcal{T}_{\Xi}$ be the collection of IVCTSFNs. Then, the IVCTSFHWA operator is mapping $T^{\mathbb{m}} \longrightarrow T$ defined as:

$$\boldsymbol{IVCTSFHWA}\,(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots \mathcal{T}_{\mathbb{m}}) = \sum_{\Xi=1}^{\mathbb{m}} \Gamma_{\Xi} \mathcal{T}_{\Xi}. \tag{11}$$

Theorem 1: Let T_{Ξ} be a group of IVCTSFNs. Then, the aggregated outcome using IVCTSFHWA is still an IVCTSFN, which is postulated as:

$$\boldsymbol{IVCTSFHWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\dots\mathcal{T}_{\mathbb{m}}) = \left(\begin{array}{l} \left[\begin{array}{l} \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathsf{l}}\left(1-\mathbb{m}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(1-\mathbb{m}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\vartheta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathsf{l}}\left(1-\vartheta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\vartheta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(1-\vartheta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}, \\ \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathsf{l}}\left(1-\mathbb{u}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(1-\mathbb{u}_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\beta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathsf{l}}\left(1-\beta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\beta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(1-\beta_{a\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}} \end{array}\right], \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{m}_{i\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{m}_{i\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\vartheta_{i\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\vartheta_{i\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}, \\ \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{u}_{i\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{u}_{i\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\beta_{i\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\beta_{i\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\beta_{i\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}} \end{array}\right], \\ \left[\begin{array}{l} \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{m}_{n\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{m}_{n\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\vartheta_{n\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\vartheta_{n\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}, \\ \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{u}_{n\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\mathbb{u}_{n\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathsf{l}}\left(\beta_{n\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathsf{l}}\left(1+(\mathbb{N}-1)\left(1-\beta_{n\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathsf{l}}\left(\beta_{n\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}} \end{array}\right] \end{array}\right) \quad (12)$$

Proof of Theorem 1 is provided Appendix-1.

Theorem 2: For the IVCTSFHWA operator, the following features hold true:

i. **Idempotency** – If for all $T_\Xi = T$, then:

$$\boldsymbol{IVCTSFHWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\dots\mathcal{T}_{\mathbb{m}}) = \mathcal{T}. \quad (13)$$

ii. **Boundedness** – Assume T^- and T^+. Then:

$$\mathcal{T}^- \leq \boldsymbol{IVCTSFHWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\dots\mathcal{T}_{\mathbb{m}}) \leq \mathcal{T}^+. \quad (14)$$

iii. **Monotonicity** – Let $\mathcal{T}_\Xi$ and P_Ξ be two collections of IVCTSFNs, such that $\mathcal{T}_\Xi \leq P_\Xi, \forall \Xi$. Then:

$$\boldsymbol{IVCTSFHWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\dots\mathcal{T}_{\mathbb{m}}) \leq \boldsymbol{IVCTSFHWA}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3,\dots\boldsymbol{P}_{\mathbb{m}}). \quad (15)$$

Proof of Theorem 2 is straightforward and therefore omitted.

The IVCTSFHWA operator first assigns the weights to IVCTSFNs and then determines their weighted aggregated value. Now, we articulate another operator termed as IVCTSFHOWA operator, which first organizes the IVCTSFNs in descending order and then allocates weights to their ordered positions. This operator is then implemented to aggregate the ordered weighted IVCTSFNs.

Definition 6: Let T_{Ξ} be a group of IVCTSFNs. We can define the IVCTSFHOWA operator as:

$$\boldsymbol{IVCTSFHOWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathfrak{l}}\Gamma_{\Xi}\,\mathcal{T}_{\sigma(\Xi)}, \tag{16}$$

where $(\sigma(1),\sigma(2),\ldots,\sigma(\mathbb{m}))$ is a permutation such that $\mathcal{T}_{\sigma(\Xi-1)}\geq\mathcal{T}_{\sigma(\Xi)},\forall\Xi$.

Theorem 3: The aggregated outcomes of the collection of IVCTSFNs T_{Ξ} by the IVCTSFHOWA operator is again an IVCTSFN and is postulated as:

$$\boldsymbol{IVCTSFHOWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathfrak{l}}\Gamma_{\Xi}\,\mathcal{T}_{\sigma(\Xi)}$$

$$=\left(\begin{array}{c}
\left[\begin{array}{c}
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{c}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{c}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\cdot e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]
\end{array}\right) \tag{17}$$

Proof of Theorem 3 is analogous to proof of Theorem 1.

The IVCTSFHOWA operator possesses the axioms of idempotency, monotonicity, and boundedness. The IVCTSFHWA operator weighs only IVCTSFNs, while the IVCTSFHOWA operator weighs just the ordered positions of IVCTSFNs. Consequently, weights express different aspects of the IVCTSFHWA and LDFHOWA operators. However, one of the operators, as well as the other operators, considers just one of them. To circumvent this shortcoming, we interpret the IVCTSFHHA operator as a generalization of both IVCTSFHWA and IVCTSFHOWA operators. This operator weighs all of the given IVCTSFNs and their appropriate order positions.

Definition 7: Suppose T_{Ξ} be a collection of IVCTSFNs. Then, the IVCTSFHHA operator is characterized as:

$$\boldsymbol{IVCTSFHHA}(\boldsymbol{\mathcal{T}_1},\boldsymbol{\mathcal{T}_2},\boldsymbol{\mathcal{T}_3},\dots\boldsymbol{\mathcal{T}_{\mathbb{m}}}) = \sum_{\Xi=1}^{\mathbb{m}} \boldsymbol{\Gamma_{\Xi}}\, \dot{\boldsymbol{\mathcal{T}}}_{\sigma(\Xi)}, \tag{18}$$

where $\dot{\mathcal{T}}_{\sigma(\Xi)}$ is the Ξ^{th} greatest element of the IVCTSF arguments $\dot{\mathcal{T}}_{\Xi}(\dot{\mathcal{T}}_{\Xi} = (kw_{\Xi})T)$ and $(\sigma(1),\sigma(2),\dots,\sigma(\mathbb{m}))$ is a permutation such that $\mathcal{T}_{\sigma(\Xi-1)} \geq \mathcal{T}_{\sigma(\Xi)}, \forall\Xi$. Here, $W = (w_1, w_2, \dots, w_{\mathbb{m}})$ is a weight vector of IVCTSFNs T_{Ξ} with $w_{\Xi} > 0$ and $w_1 + w_2 + \dots + w_{\mathbb{m}} = 1$.

Theorem 4**:** Let T_{Ξ} be a gathering of IVCTSFNs. Then, the IVCTSFHHA operator is defined as:

$$\boldsymbol{IVCTSFHHA}(\boldsymbol{\mathcal{T}_1},\boldsymbol{\mathcal{T}_2},\boldsymbol{\mathcal{T}_3},\dots\boldsymbol{\mathcal{T}_{\mathbb{m}}})$$

$$\left(\begin{array}{c}
\left[\begin{array}{c}
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{c}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\vartheta}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{i_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{c}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\vartheta}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{n_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]
\end{array}\right). \tag{19}$$

Proof of Theorem 4 is straightforward.

5. Geometric Aggregation Operators of IVCTSFNs

In this section, using Hamacher operational laws, we establish a class of geometric AOs for IVCTSFNs, namely the IVCTSFHWG, IVCTSFHOWG, and IVCTSFHHWG operators and studied some of their basic features.

Definition 8: For a group of IVCTSFNs T_{Ξ} the IVCTSFHWG operator is portrayed as:

$$\boldsymbol{IVCTSFHWG}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathbb{l}}\mathcal{T}_{\Xi}^{\Gamma_{\Xi}}. \tag{20}$$

Theorem 5: Consider T_{Ξ} be IVCTSFNs. The IVCTSFHWG operator is described as:

$$\boldsymbol{IVCTSFHWG}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})$$

$$=\left(\begin{array}{l}\left[\begin{array}{l}\dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{l}}\left(\mathbb{m}_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{a_{\Xi}}^{\mathbb{Q}}\right)\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(\mathbb{m}_{a_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{l}}\left(\vartheta_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)(1-\vartheta_{a_{\Xi}})\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(\vartheta_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}}},\\ \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{l}}\left(\mathbb{u}_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{a_{\Xi}}^{\mathbb{Q}}\right)\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(\mathbb{u}_{a_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{l}}\left(\beta_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)(1-\beta_{a_{\Xi}})\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(\beta_{a_{\Xi}}\right)^{\Gamma_{\Xi}}}}}\end{array}\right],\\ \left[\begin{array}{l}\sqrt[\mathbb{Q}]{\dfrac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{m}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{m}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\vartheta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\vartheta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\vartheta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\vartheta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}},\\ \sqrt[\mathbb{Q}]{\dfrac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{u}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{u}_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\beta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\beta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\beta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\beta_{\mathrm{i}_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}}\end{array}\right],\\ \left[\begin{array}{l}\sqrt[\mathbb{Q}]{\dfrac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{m}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{m}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{m}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\vartheta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\vartheta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\vartheta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\vartheta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}},\\ \sqrt[\mathbb{Q}]{\dfrac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{u}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\mathbb{u}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\mathbb{u}_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\beta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}-\prod_{\Xi=1}^{\mathbb{l}}\left(1-\beta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}{\prod_{\Xi=1}^{\mathbb{l}}\left(1+(\mathbb{N}-1)\beta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{l}}\left(1-\beta_{n_{\Xi}}^{\mathbb{Q}}\right)^{\Gamma_{\Xi}}}}}\end{array}\right]\end{array}\right). \tag{21}$$

Proof of Theorem 5 is analogous to proof of Theorem 1.

Definition 9: Consider T_{Ξ} be a collection of IVCTSFNs. Then, the IVCTSFHOWG operator is postulated as:

$$\boldsymbol{IVCTSFHOWG}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathbb{l}}\mathcal{T}_{\sigma(\Xi)}^{\Gamma_{\Xi}}, \tag{22}$$

where $(\sigma(1),\sigma(2),\ldots,\sigma(\mathbb{m}))$ is a permutation such that $\mathcal{T}_{\sigma(\Xi-1)}\geq\mathcal{T}_{\sigma(\Xi)},\forall\Xi$.

Theorem 6: Consider T_{Ξ} be a family of IVCTSFNs. Then, the IVCTSFHOWG operator is given as:

$$\boldsymbol{IVCTSFHOWv}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathfrak{l}}\mathcal{T}_{\sigma(\Xi)}^{\Gamma_\Xi}$$

$$=\left(\begin{matrix}\left[\begin{matrix}\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{as_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{m}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\mathbb{u}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\beta_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right],\\ \left[\begin{matrix}\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right],\\ \left[\begin{matrix}\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{m}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\vartheta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\mathbb{u}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\beta_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right]\end{matrix}\right). \quad (23)$$

Definition 10: Suppose that T_Ξ be a collection of IVCTSFNs. Then, IVCTSFHHG is postulated as:

$$\boldsymbol{IVCTSFHHv}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})=\sum_{\Xi=1}^{\mathfrak{l}}\dot{\mathcal{T}}_{\sigma(\Xi)}^{\Gamma_\Xi}. \quad (24)$$

Theorem 7: Consider T_Ξ a group of IVCTSFNs. Then, IVCTSFHHG is characterized as:

$$\boldsymbol{IVCTSFHHv}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_{\mathbb{m}})$$

$$\left(\begin{matrix}\left[\begin{matrix}\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{m}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\vartheta_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\vartheta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\mathbb{u}}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{a_{\sigma(\Xi)}}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\left(1-\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(\dot{\beta}_{a_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right]\\ \left[\begin{matrix}\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{i_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right]\\ \left[\begin{matrix}\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{m}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\vartheta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\ \sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\mathbb{u}}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{\mathbb{m}}\left(1+(\mathbb{N}-1)\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{\mathbb{m}}\left(1-\dot{\beta}_{n_{\sigma(\Xi)}}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}\end{matrix}\right]\end{matrix}\right) \quad (25)$$

6. Application of MADM

In this segment, we implemented the proposed operators to construct a MADM scheme under the environment of the IVCTSFS setting. Let $B = \{B_1, B_2, \ldots, B_k\}$ be an assemblage of alternatives and $\upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_\Xi\}$ be the set attributes. Assume that $\Gamma_\Xi = (\Gamma_1, \Gamma_2, \ldots, \Gamma_{\mathbb{m}})^T$ be the associated weight vector of attributes, such that $\Gamma_\Xi > 0$ and $\sum_1^{\mathbb{m}} \Gamma_\Xi = 1$.

Presume that $D_{k\times\Xi} = (T)_{k\times\Xi} = \begin{pmatrix} [\mathbb{m}_a(\mathfrak{X}).e^{2\pi i\vartheta_a(\mathfrak{X})}, \mathbb{u}_a(\mathfrak{X}).e^{2\pi i\beta_a(\mathfrak{X})}], \\ [\mathbb{m}_i(\mathfrak{X}).e^{2\pi i\vartheta_i(\mathfrak{X})}, \mathbb{u}_i(\mathfrak{X}).e^{2\pi i\beta_i(\mathfrak{X})}], \\ [\mathbb{m}_n(\mathfrak{X}).e^{2\pi i\vartheta_n(\mathfrak{X})}, \mathbb{u}_n(\mathfrak{X}).e^{2\pi i\beta_n(\mathfrak{X})}] \end{pmatrix}_{k\times\Xi}$ be the decision matrix of IVCTSFNs provided by an expert to evaluate the given alternatives under given attributes. Considering these observations, the systematic process of designed MADM scheme is outlined as follows:

Step 1 – Structure the given information in the decision matrix form of IVCTSFNs.

Step 2 – Evaluate the normalized matrix $\widehat{N} = (\widehat{T})_{k\times\Xi}$ according to the following transformation:

$$\widehat{\boldsymbol{T}} = \begin{cases} \boldsymbol{T} \text{ for benifit attribute} \\ \boldsymbol{T}^c \text{ for cost attribute} \end{cases}. \tag{26}$$

Step 3 – Utilize the proposed AOs to aggregate data in the form of IVCTSFNs.

Step 4 – Compute the score values of each alternative using the following formula:

$$\mathbb{S}(\boldsymbol{T}) = \frac{((\mathbb{m}_a+\boldsymbol{\vartheta}_a+\mathbb{u}_a+\boldsymbol{\beta}_a)-(\mathbb{m}_i+\boldsymbol{\vartheta}_i+\mathbb{u}_i+\boldsymbol{\beta}_i)-(\mathbb{m}_n+\boldsymbol{\vartheta}_n+\mathbb{u}_n+\boldsymbol{\beta}_n))}{\boldsymbol{6}}. \tag{27}$$

Step 5 – Rank the alternatives based on their score values. The greater the score value, the better the alternative is.

6.1. Numerical Example

This example helps us apply the proposed way to deal with the MADM issue. The government made a medical college project in the city and selected a suitable place for this project. Medical colleges are essential institutions that play a multifaceted role in training healthcare professionals, advancing medical research, providing healthcare services, and contributing to the overall health and well-being of communities. They are foundational elements of healthcare systems, supporting the delivery of quality care and the pursuit of medical knowledge. The benefits of medical colleges are:

i. *Training healthcare professionals* **–** To educate and train doctors, nurses, and allied health professionals, ensuring a skilled workforce for healthcare delivery.
ii. *Advancing medical research* – Researching to improve medical knowledge, treatments, and patient outcomes.
iii. *Addressing healthcare disparities* – Alleviating healthcare disparities by producing professionals attuned to diverse community needs.
iv. *Providing specialized medical services* – Offering specialized medical care and treatments through associated hospitals and clinics.

v. *Meeting community healthcare needs* – Medical colleges can engage in international collaborations, sharing expertise, research findings, and best practices with the global healthcare community.

In essence, medical colleges are critical for training, research, and healthcare delivery, ensuring a healthy and skilled workforce for the benefit of society. Ultimately, the suitability of an area for a medical college is a complex decision that takes into account the unique characteristics and needs of the region, to improve healthcare services and education for the benefit of the local population. The government wants to select a city to make medical college due to some basic's cause. This city's population is larger than other cities. The government chooses four cities for this project. The branch of wellbeing needs to choose one city B_{Ξ} $(1 \leq \Xi \leq 4)$ in view of certain criteria (υ_1), (υ_2), (υ_3), and (υ_4). The weights are $\Gamma_{\Xi} = (0.1, 0.2, 0.3, 0.4)^T$.

Step 1 – The data provided by an expert is organized in the from of IVCTFNs (Table 1).

Table 1
Decision matrix

	υ_1	υ_2
B_1	$\left([0.3, 0.4]e^{2\pi i[0.2,0.3]}, [0.2, 0.3]e^{2\pi i[0.3,0.4]}, [0.2, 0.4]e^{2\pi i[0.2,0.4]}\right)$	$\left([0.3, 0.5]e^{2\pi i[0.2,0.3]}, [0.32, 0.35]e^{2\pi i[0.22,0.25]}, [0.1, 0.3]e^{2\pi i[0.2,0.3]}\right)$
B_2	$\left([0.2, 0.3]e^{2\pi i[0.1,0.4]}, [0.1, 0.3]e^{2\pi i[0.4,0.5]}, [0.3, 0.4]e^{2\pi i[0.2,0.3]}\right)$	$\left([0.3, 0.4]e^{2\pi i[0.3,0.4]}, [0.22, 0.34]e^{2\pi i[0.32,0.46]}, [0.2, 0.4]e^{2\pi i[0.3,0.4]}\right)$
B_3	$\left([0.1, 0.4]e^{2\pi i[0.2,0.4]}, [0.3, 0.4]e^{2\pi i[0.1,0.3]}, [0.2, 0.3]e^{2\pi i[0.1,0.2]}\right)$	$\left([0.2, 0.4]e^{2\pi i[0.4,0.5]}, [0.34, 0.35]e^{2\pi i[0.25,0.44]}, [0.2, 0.4]e^{2\pi i[0.2,0.4]}\right)$
B_4	$\left([0.2, 0.6]e^{2\pi i[0.3,0.5]}, [0.4, 0.5]e^{2\pi i[0.2,0.4]}, [0.3, 0.4]e^{2\pi i[0.3,0.4]}\right)$	$\left([0.1, 0.2]e^{2\pi i[0.3,0.4]}, [0.21, 0.25]e^{2\pi i[0.17,0.45]}, [0.3, 0.4]e^{2\pi i[0.1,0.2]}\right)$
	υ_3	υ_4
B_1	$\left([0.1, 0.2]e^{2\pi i[0.2,0.3]}, [0.22, 0.35]e^{2\pi i[0.24,0.28]}, [0.2, 0.3]e^{2\pi i[0.3,0.4]}\right)$	$\left([0.1, 0.3]e^{2\pi i[0.2,0.3]}, [0.24, 0.25]e^{2\pi i[0.22,0.25]}, [0.3, 0.4]e^{2\pi i[0.1,0.4]}\right)$
B_2	$\left([0.3, 0.4]e^{2\pi i[0.3,0.4]}, [0.24, 0.33]e^{2\pi i[0.34,0.35]}, [0.3, 0.4]e^{2\pi i[0.2,0.3]}\right)$	$\left([0.2, 0.3]e^{2\pi i[0.3,0.4]}, [0.22, 0.35]e^{2\pi i[0.32,0.35]}, [0.3, 0.5]e^{2\pi i[0.2,0.5]}\right)$
B_3	$\left([0.4, 0.5]e^{2\pi i[0.1,0.2]}, [0.17, 0.35]e^{2\pi i[0.22,0.25]}, [0.1, 0.2]e^{2\pi i[0.2,0.4]}\right)$	$\left([0.3, 0.4]e^{2\pi i[0.4,0.5]}, [0.21, 0.25]e^{2\pi i[0.23,03.4]}, [0.2, 0.3]e^{2\pi i[0.2,0.4]}\right)$
B_4	$\left([0.2, 0.4]e^{2\pi i[0.1,0.3]}, [0.19, 0.25]e^{2\pi i[0.21,0.27]}, [0.2, 0.3]e^{2\pi i[0.1,0.2]}\right)$	$\left([0.4, 0.5]e^{2\pi i[0.4,0.5]}, [0.16, 0.22]e^{2\pi i[0.22,0.33]}, [0.2, 0.4]e^{2\pi i[0.3,0.4]}\right)$

Step 2 – There is no need for normalization since there is no cost attribute.

Step 3 – The aggregated values (fixing $\mathbb{Q} = 4, \mathbb{N} = 3$) via the IVCTSFHWA and IVCTSFHWG operators are displayed in Table 2.

Table 2
Aggregated values

	IVCTSFHWA	IVCTSFHWG
B_1	$\left([0.0893, 0.1279]e^{2\pi i[0.0895,0.1186]}, [0.0314, 0.039]e^{2\pi i[0.0301,0.0355]}, [0.0261, 0.0456]e^{2\pi i[0.0218,0.0497]}\right)$	$\left([0.0178, 0.0402]e^{2\pi i[0.0255,0.0391]}, [0.1036, 0.1223]e^{2\pi i[0.1016,0.1138]}, [0.0871, 0.1308]e^{2\pi i[0.0921,0.1383]}\right)$
B_2	$\left([0.1059, 0.1330]e^{2\pi i[0.1052,0.14445]}, [0.0266, 0.0441]e^{2\pi i[0.0439,0.0511]}, [0.0359, 0.0573]e^{2\pi i[0.0278,0.0511]}\right)$	$\left([0.0316, 0.0456]e^{2\pi i[0.0347,0.0527]}, [0.0876, 0.1276]e^{2\pi i[0.1287,0.1437]}, [0.1108, 0.1526]e^{2\pi i[0.0976,0.1419]}\right)$
B_3	$\left([0.10561, 0.1516]e^{2\pi i[0.1127,0.144445]}, [0.0291, 0.0409]e^{2\pi i[0.0271,0.0424]}, [0.1, 0.2]e^{2\pi i[0.2,0.4]}\right)$	$\left([0.0349, 0.0562]e^{2\pi i[0.0321,0.0495]}, [0.1044, 0.1264]e^{2\pi i[0.0892,0.1270]}, [0.0812, 0.1174]e^{2\pi i[0.0810,0.1358]}\right)$
B_4	$\left([0.0960, 0.1463]e^{2\pi i[0.1112,0.1489]}, [0.0252, 0.0335]e^{2\pi i[0.0260,0.0446]}, [0.0206, 0.0484]e^{2\pi i[0.0221,0.037]}\right)$	$\left([0.0295, 0.0530]e^{2\pi i[0.0316,0.0544]}, [0.0971, 0.11301]e^{2\pi i[0.0890,0.1324]}, [0.1032, 0.1371]e^{2\pi i[0.0846,0.1155]}\right)$

Step 4 – The score values are calculated in Table 3.

Table 3
Score values

	IVCTSFHWA	IVCTSFHWG
$\mathbb{S}(\boldsymbol{B_1})$	$\mathbf{0.0509}$	$\mathbf{-0.3807}$
$\mathbb{S}(\boldsymbol{B_2})$	$\mathbf{0.0423}$	$\mathbf{-0.3839}$
$\mathbb{S}(\boldsymbol{B_3})$	$\mathbf{0.0818}$	$\mathbf{-0.3489}$
$\mathbb{S}(\boldsymbol{B_4})$	$\mathbf{0.0806}$	$\mathbf{-0.3515}$

Step 5 – Based on the score values, the outcomes through the IVCTSFHWA and IVCTSFHWG operators are obtained as $B_3 \succ B_4 \succ B_1 \succ B_2$ and $B_3 \succ B_4 \succ B_1 \succ B_2$, respectively.

7. Discussion and Comparison

The present section conducts a comparative sensitivity analysis and comparison of the designed AOs to confirm the validity and supremacy of our recommended technique and operators.

7.1. Sensitivity Analysis

Depending on the decision-maker choices, multiple values might be allocated to the Hamacher parameter Γ_{Ξ}. To scrutinize the influence of the parameter Γ_{Ξ} on the recommended MADM strategy, we accomplish an analysis using numerous values of Γ_{Ξ}. The overall score values and the ranking results utilizing the IVCTSFHWA and IVCTSFHWG operators related to these values of Γ_{Ξ} are summarized in Table 4 and Table 5, respectively.

Table 4
Impact of Γ_{Ξ} under the IVCTSFHWA operator

Inputs of $\boldsymbol{\Gamma_{\Xi}}$	Score values				Rankings
	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$	
2	$\mathbf{-0.2365}$	$\mathbf{-0.2457}$	$\mathbf{-0.2203}$	$\mathbf{-0.2226}$	$\boldsymbol{B_3 \succ B_4 \succ B_2 \succ B_1}$
4	$\mathbf{-0.1878}$	$\mathbf{-0.0901}$	$\mathbf{-0.1662}$	$\mathbf{-0.1682}$	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
5	$\mathbf{-0.1693}$	$\mathbf{-0.178}$	$\mathbf{-0.1459}$	$\mathbf{-0.1478}$	$\boldsymbol{B_4 \succ B_3 \succ B_1 \succ B_2}$
7	$\mathbf{-0.1396}$	$\mathbf{-0.1474}$	$\mathbf{-0.1135}$	$\mathbf{-0.1154}$	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
8	$\mathbf{-0.1274}$	$\mathbf{-0.1346}$	$\mathbf{-0.1009}$	$\mathbf{-0.1021}$	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
9	$\mathbf{-0.1164}$	$\mathbf{-0.1231}$	$\mathbf{-0.0884}$	$\mathbf{-0.0902}$	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
12	$\mathbf{-0.0887}$	$\mathbf{-0.0941}$	$\mathbf{-0.0587}$	$\mathbf{-0.0605}$	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$

Table 5
Impact of Γ_{Ξ} under the IVCTSFHWG operator

Inputs of $\boldsymbol{\Gamma_{\Xi}}$	$\boldsymbol{Score\ values}$				$\boldsymbol{Ranking}$
	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$	
2	0.0646	0.0649	0.084	0.0825	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
4	0.0129	0.0113	0.0377	0.0362	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
5	-0.0071	-0.0096	0.0192	0.0177	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
7	-0.0404	-0.0445	-0.0118	-0.0134	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
8	-0.0546	-0.0595	-0.0253	-0.0269	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
9	-0.0677	-0.0731	-0.0378	-0.03937	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$
12	-0.1017	-0.1086	-0.0705	-0.0720	$\boldsymbol{B_3 \succ B_4 \succ B_1 \succ B_2}$

We can see from Table 4 and Table 5 that the overall ranking of alternatives remains consistent, with B_3 as the optimal alternative. As a result, the devised technique has a high degree of stability w.r.t. the Γ_{Ξ} inputs.

7.2. Comparative Analysis

In this section, we compare our proposed scheme with several AOs to prove the efficiency and superiority of the recommended method. Ali *et al.* [36] initiated a MADM approach using T-spherical fuzzy AOs. Ullah *et al.* [31] studied T-spherical fuzzy Hamacher weighted averaging (T-SFHWA) and T-spherical fuzzy Hamacher weighted geometric (T-SFHWG) operators with application to the evaluation of the performance of search and rescue robots. We resolved the previously stated MADM problem by employing the methods of [32, 37], and the outcomes are exhibited in Table 6.

Table 6
Ranking of the alternatives

Operators	Score values				Ranking result
	B_1	B_2	B_3	B_4	
$IVCTSFHWA$	0.0509	0.0423	0.0818	0.0806	$B_3 \succ B_4 \succ B_1 \succ B_2$
$IVCTSFHWG$	-0.3807	-0.3839	-0.3489	-0.3515	$B_3 \succ B_4 \succ B_1 \succ B_2$
CTSFWA [36]	-0.1164	-0.1231	-0.0884	-0.0902	$B_3 \succ B_4 \succ B_1 \succ B_2$
$CTSFWG$ [36]	-0.0677	-0.0731	-0.0378	-0.03937	$B_3 \succ B_4 \succ B_1 \succ B_2$
T − SFHWA [31]	-0.0887	-0.0941	-0.0587	-0.0605	$B_3 \succ B_4 \succ B_1 \succ B_2$
T − SFHWG [31]	-0.1017	-0.1086	-0.0705	-0.0720	$B_3 \succ B_4 \succ B_1 \succ B_2$

According to Table 6, it becomes evident that the ranking orders acquired through these methods and the developed scheme are identical. This reveals that the designed MADM approach is valid. Also, the results are graphically plotted in Figure 1.

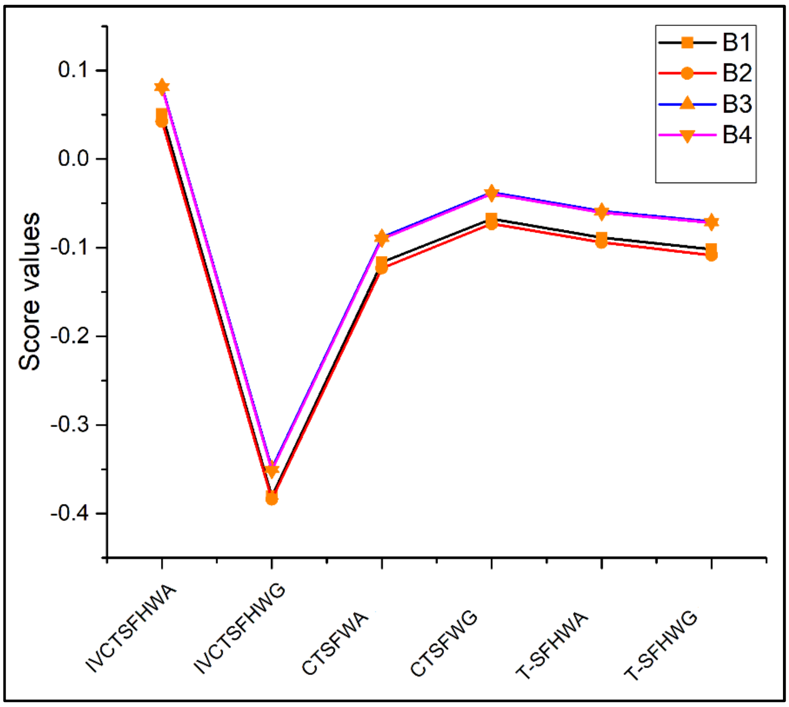

Fig. 1. Ranking of different methods

8. Conclusions

The IVCTSFS model is a generalization of TSFS and CFS. The Hamacher AOs are more flexible and powerful in many issues involving uncertainty, as they incorporate a regulatory parameter that can play a vital role in managing extreme values. Having a generic nature, both IVCTSFSs and Hamacher AOs require further study to be conducted. In this article, we established several Hamacher operational rules for IVCTSFNs. By using the Hamacher operational laws, we constructed a class of averaging and geometric AOs, namely the IVCTSFHWA, IVCTSFHOWA, IVCTSFHHWA, IVCTSFHWG, IVCTSFOWG, and IVCTSFHHWG operators. We also provided several cardinal features like idempotency, monotonicity, and boundedness. Meanwhile, we have designed a MADM method using the proposed operator.

To illustrate the proficiency and legitimacy of the developed method, we considered a MADM problem and solved it within the framed method. The influence of Hamacher variable parameters on the decision-making process is scrutinized, and the stability of ranking outcomes is deliberated. Lastly, a comparative analysis of rankings obtained using the projected and prevailing AOs underscored the significance of the designed methodology.

In our future work, we will extend our proposed work within the framework of complex neutrosophic hesitant fuzzy sets and spherical linear Diophantine fuzzy sets.

Appendix-1: Proof of Theorem 1

We will use the mathematical induction method to prove the desired result.

For $\mathrm{l} = 2$, we have:

$$\boldsymbol{\Gamma_1 \mathcal{T}_1 \oplus \Gamma_2 \mathcal{T}_2}$$

$$= \left(\begin{array}{l} \left[\begin{array}{l} \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-\left(1-\mathbb{m}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-(\mathbb{N}-1)\left(1-\mathbb{m}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-\left(1-\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}{\left(1+(\mathbb{N}-1)\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-(\mathbb{N}-1)\left(1-\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}}, \\ \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-\left(1-\mathbb{u}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-(\mathbb{N}-1)\left(1-\mathbb{u}_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \sqrt[\mathbb{Q}]{\dfrac{\left(1+(\mathbb{N}-1)\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-\left(1-\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}{\left(1+(\mathbb{N}-1)\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}-(\mathbb{N}-1)\left(1-\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}} \end{array} \right], \\ \left[\begin{array}{l} \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{i_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\mathbb{m}_{i_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{i_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\vartheta_{i_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}}, \\ \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{i_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\mathbb{u}_{i_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{i_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{i_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\beta_{i_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}} \end{array} \right], \\ \left[\begin{array}{l} \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{n_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\mathbb{m}_{n_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{n_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\vartheta_{n_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}}, \\ \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{n_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\mathbb{u}_{n_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}.\,e^{2\pi i \dfrac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{n_1}\right)^{\Gamma_1}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{n_1}^{\mathbb{Q}}\right)\right)^{\Gamma_1}+(\mathbb{N}-1)\left(\beta_{n_1}^{\mathbb{Q}}\right)^{\Gamma_1}}}} \end{array} \right] \end{array} \right)$$

$$
\oplus\left(\begin{array}{l}
\left[\begin{array}{l}
\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}-\left(1-\mathbb{m}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}}{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}-(\mathbb{N}-1)\left(1-\mathbb{m}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}-\left(1-\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}}{\left(1+(\mathbb{N}-1)\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}-(\mathbb{N}-1)\left(1-\vartheta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}}}},\\
\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}-\left(1-\mathbb{u}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}}{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}-(\mathbb{N}-1)\left(1-\mathbb{u}_{a_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}-\left(1-\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}}{\left(1+(\mathbb{N}-1)\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}-(\mathbb{N}-1)\left(1-\beta_{a_1}^{\mathbb{Q}}\right)^{\Gamma_2}}}}
\end{array}\right]\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{i_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\mathbb{m}_{i_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{i_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\vartheta_{i_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{i_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\mathbb{u}_{i_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{i_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{i_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\beta_{i_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}}
\end{array}\right]\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{n_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\mathbb{m}_{n_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{n_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\vartheta_{n_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{n_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\mathbb{u}_{n_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{n_2}\right)^{\Gamma_2}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{n_2}^{\mathbb{Q}}\right)\right)^{\Gamma_2}+(\mathbb{N}-1)\left(\beta_{n_2}^{\mathbb{Q}}\right)^{\Gamma_2}}}}
\end{array}\right]
\end{array}\right)
$$

$$
=\left(\begin{array}{l}
\left[\begin{array}{l}
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{2}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{2}\left(1-\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{2}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{2}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\mathbb{m}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\vartheta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\vartheta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\mathbb{u}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\beta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\beta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\beta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right],\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\mathbb{m}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\vartheta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\vartheta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\mathbb{u}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{2}\left(\beta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{2}\left(1+(\mathbb{N}-1)\left(1-\beta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{2}\left(\beta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]
\end{array}\right).
$$

This reveals that for $\mathfrak{l} = 2$ the result is valid.
Now, we can prove the result is true for $\mathfrak{l} = k \oplus 1$and $\mathfrak{l} = k$.

$$IVCTSFHWA(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\ldots\mathcal{T}_k)\oplus\mathcal{T}_{k+1}$$

$$
\begin{aligned}
=&\left(\begin{matrix}
\left[\begin{matrix}
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k}\left(1-\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(1-\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{matrix}\right],\\
\left[\begin{matrix}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\mathbb{m}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\vartheta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\vartheta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\mathbb{u}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\beta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\beta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\beta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{matrix}\right],\\
\left[\begin{matrix}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\mathbb{m}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\vartheta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\vartheta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\mathbb{u}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k}\left(\beta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k}\left(1+(\mathbb{N}-1)\left(1-\beta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k}\left(\beta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{matrix}\right]
\end{matrix}\right)\\
\oplus&\left(\begin{matrix}
\left[\begin{matrix}
\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}-\left(1-\mathbb{m}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}{\left(1+(\mathbb{N}-1)\mathbb{m}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(1-\mathbb{m}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\vartheta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}-\left(1-\vartheta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}{\left(1+(\mathbb{N}-1)\vartheta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(1-\vartheta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}},\\
\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}-\left(1-\mathbb{u}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}{\left(1+(\mathbb{N}-1)\mathbb{u}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(1-\mathbb{u}_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\left(1+(\mathbb{N}-1)\beta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}-\left(1-\beta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}{\left(1+(\mathbb{N}-1)\beta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(1-\beta_{a_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}}
\end{matrix}\right],\\
\left[\begin{matrix}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{i_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\mathbb{m}_{i_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{i_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\vartheta_{i_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{i_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\mathbb{u}_{i_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{i_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{i_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\beta_{i_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}}
\end{matrix}\right],\\
\left[\begin{matrix}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{m}_{n_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\mathbb{m}_{n_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\vartheta_{n_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\vartheta_{n_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\mathbb{u}_{n_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\mathbb{u}_{n_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}.e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\left(\beta_{n_{k+1}}\right)^{\Gamma_{k+1}}}{\sqrt[\mathbb{Q}]{\left(1+(\mathbb{N}-1)\left(1-\beta_{n_{k+1}}^{\mathbb{Q}}\right)\right)^{\Gamma_{k+1}}+(\mathbb{N}-1)\left(\beta_{n_{k+1}}^{\mathbb{Q}}\right)^{\Gamma_{k+1}}}}}
\end{matrix}\right]
\end{matrix}\right)
\end{aligned}
$$

$$\boldsymbol{IVCTSFHWA}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\dots\mathcal{T}_{k+1}) =$$

$$\left(\begin{array}{l}
\left[\begin{array}{l}
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k+1}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(1-\mathbb{m}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k+1}\left(1-\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(1-\vartheta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k+1}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(1-\mathbb{u}_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\sqrt[\mathbb{Q}]{\frac{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}-\prod_{\Xi=1}^{k+1}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(1-\beta_{a_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\mathbb{m}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\mathbb{m}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\vartheta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\vartheta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\mathbb{u}_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\mathbb{u}_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\beta_{i_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\beta_{i_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\beta_{i_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]\\
\left[\begin{array}{l}
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\mathbb{m}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\mathbb{m}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\vartheta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\vartheta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\vartheta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}},\\
\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\mathbb{u}_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\mathbb{u}_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}\,.\,e^{2\pi i\frac{\sqrt[\mathbb{Q}]{\mathbb{N}}\prod_{\Xi=1}^{k+1}\left(\beta_{n_\Xi}\right)^{\Gamma_\Xi}}{\sqrt[\mathbb{Q}]{\prod_{\Xi=1}^{k+1}\left(1+(\mathbb{N}-1)\left(1-\beta_{n_\Xi}^{\mathbb{Q}}\right)\right)^{\Gamma_\Xi}+(\mathbb{N}-1)\prod_{\Xi=1}^{k+1}\left(\beta_{n_\Xi}^{\mathbb{Q}}\right)^{\Gamma_\Xi}}}}
\end{array}\right]
\end{array}\right)$$

This show that the result is valid for $\mathfrak{l} = k \oplus 1$. So, according to mathematical induction, it is proven that the result is true for all positive integers $\mathfrak{l}$.

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Conflicts of Interest
The author declares no conflicts of interest.

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